Finite time convergence analysis for "Twisting" controller via a strict Lyapunov function

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***Resumen*— A second order sliding mode controller, the so-called "Twisting" algorithm is under study. A non-smooth strict Lyapunov function is proposed, so global finite time stability for this algorithm can be proved, even in the case when it is affected by bounded external perturbations. The strict Lyapunov function gives the possibility to estimate an upper bound for the time convergence of the trajectories of the system to the equilibrium point. A linear compensator is added to the twisting algorithm so linear increasing perturbations can be compensated. Indeed, finite time stability and an estimation for time convergence for the quasi-homogeneous synthesis is shown.**

Keywords: Sliding mode; variable structure systems; stability analysis.

I. Introduction

A. State of art

In last years, second order sliding mode algorithms become very important for Variable Structure Systems (VSS) theory. One of the first algorithms is the well-known "Twisting" algorithm (see (Emelyanov, Korovin and Levantovsky, 1986)). The twisting algorithm is given by

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\alpha\mathrm{sgn}(x) - \beta\mathrm{sgn}(y) + \delta\end{aligned}$$

where x, $y \in \mathbb{R}$ are scalar state variables and $|\delta| \leq M$ is a bounded term of uncertainty. Notice that for the nominal case of (1), when $\delta = 0$, we have a double integrator case. In this case, the algorithm is globally uniformly finite time stable if the inequality

$$\alpha > \beta > 0 \tag{1}$$

is satisfied (Orlov, 2006). It is well known that no sliding motion appears on the axes $x = 0$ and $y = 0$ except the origin $x = y = 0$, which proves to be the only equilibrium point of the switched system (1). In (Levant, 1993), a geometrical proof for convergence for twisting algorithm is presented. Even, the estimation of the time convergence is presented, the calculation of this feature seems to be very difficult using this kind of analysis. In (Levant, 1993), the stability proof for the non-perturbed algorithm (1) is based on homogeneity properties and the following weak Lyapunov function

$$V = \alpha|x| + \frac{1}{2}y^2, \tag{2}$$

where the time derivative of (2) along the trajectories of (1) is

$$\dot{V} = -\beta|y|. \tag{3}$$

Using this weak Lyapunov function (2), only stability can be guaranteed. Indeed, homogeneity properties are needed to prove finite time stability (see (Orlov, 2005)). There are another works that use homogeneity approach in order to prove finite time convergence (see (Bacciotti and Rosier, 2001), (Hong, Huang and Xu, 2001), (Shtessel, Shkolnikov and Levant, 2007)), but with this kind of approach an upper bound for time convergence can not be calculated. A strict Lyapunov design and an estimation for reaching time, using Zubov method, is presented in (Polyakov and Poznyak, 2009). For design control purposes, this methodology becomes difficult. Indeed, it requires some handicraft techniques, like fixing the function discontinuities. The quasi-homogeneous synthesis is given by system (1) and a linear compensator, so the system is given by

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\alpha\mathrm{sgn}(x) - \beta\mathrm{sgn}(y) - hx - py + \delta\end{aligned}$$

where $h, p > 0$. In (Orlov, 2005), global uniform finite time stability is shown for this algorithm, using homogeneity principles.

Summing up, some of the disadvantages for the above methodology is listed on the following remarks:

1. Geometric proof is only a semi-global one and the estimation for reaching time is difficult to obtain, also does not exists for the system (4),
2. the homogeneity proof is unable to give an estimation for time convergence,
3. strict Lyapunov functions, using Zubov method, are very complicated for control design purposes and has not been constructed for the system (4).

B. Main contributions

The main contributions of this work can be listed as follows:

- A strict Lyapunov function design, for the twisting algorithm and the quasi-homogeneous synthesis is presented, and uniform finite time stability is shown,
- An upper bound for time convergence is calculated using these strict Lyapunov function, for the non perturbed and the perturbed case.

1) Structure: This paper has the following structure: in section (II) a strict Lyapunov function is proposed so global finite time stability for twisting algorithm can be proved and an upper bound for time convergence is calculated. In order to present the information clearly, the proofs of this section are omitted, and in the next section the proof for finite time stability for a general case will be presented. In section (III) a finite time stability analysis for twisting algorithm with a linear compensator is shown, and its local upper bound for time convergence also. In section (IV) numerical experiments are presented using the cart-pendulum as test bed. At section (V) the conclusions of this work are given.

II. A STRICT LYAPUNOV FUNCTION FOR STABILITY ANALYSIS OF TWISTING ALGORITHM.

A. Description of the system.

Consider the controlled system given by

$$\dot{x} = y$$
$$\dot{y} = \delta + u$$

where x and $y \in \mathbb{R}$ are scalar state variables, $|\delta| \leq M$ is a bounded term of uncertainty and $u \in \mathbb{R}$ is the so-called "twisting" control (see (Orlov, 2006))

$$u = -\alpha \text{sgn}(x) - \beta \text{sgn}(y) \tag{4}$$

where $\alpha, \beta > 0$ are control parameters. Consider that the right hand side of (4) is discontinuous, so the solutions of the system (4) are defined in Filippov sense (Filippov, 1988). The nominal system (4), the case when $\delta = 0$, is globally uniformly finite time stable if the inequality

$$\alpha > \beta > 0 \tag{5}$$

is satisfied (see (Orlov, 2006)). It is well-known that no sliding motion appears on the axes $x = 0$ and $y = 0$ except the origin $x = y = 0$, which proves to be the only equilibrium point of the switched system (4). Let

$$V(x,y) = \alpha^2\gamma_1 x^2 + \gamma_2 |x|^{\frac{3}{2}} \text{sgn}(x) y + \alpha\gamma_1 |x| y^2 + \frac{1}{4}\gamma_1 y^4 \tag{6}$$

a positive definite Lyapunov function, where $\gamma_i > 0$. This function is continuous everywhere but not differentiable for $x = 0$. It will be shown that (6) is a strict Lyapunov function (see (Bacciotti, A., and Rosier, 2001)) for the system (4), so the finite time stability will be proved. Moreover, an upper bound for time convergence for the trajectories of (4) to zero is given.

Theorem 1: Let (6) be a strict Lyapunov function for the unperturbed system (4), where

$$\gamma_2 < \frac{4\sqrt{2}}{3}\gamma_1 \beta \sqrt{\alpha} \tag{7}$$

holds for all $\gamma_i > 0$, $i = 1, 2$, and let the parameters α and β from the system (4) be such that the condition (5) is satisfied. Then the system (4) is global finite time stable around the origin with

$$t \leq \frac{12\lambda_{max}^{\frac{3}{4}}(\Theta)}{min\,(\kappa)} V^{\frac{1}{4}}(x(0), y(0)) \tag{8}$$

and $\kappa = \{\gamma_2(\alpha - \beta), \alpha\beta\gamma_1, \frac{1}{2}\gamma_2, \frac{1}{2}\gamma_1\beta\}$. as an upper bound for time convergence for the trajectories to the equilibrium point.

In the perturbed dynamics, when $\delta \neq 0$, it can be shown that the twisting algorithm is globally uniformly finite time stable spite of the external bounded perturbations.

Theorem 2: Let

$$\gamma_2 < \frac{4\sqrt{2}}{3}\gamma_1 (\beta - M) \sqrt{\alpha} \tag{9}$$

and

$$\alpha - M > \beta > M \tag{10}$$

be satisfied and the function (6) a strict Lyapunov function for the system (4). Then all the trajectories of the perturbed system (4) converge to zero in a finite time transient with a maximal duration of

$$t_{reach} \leq \frac{12\lambda_{max}^{\frac{3}{4}}(\Theta)}{min\,(\kappa_\delta)} V^{\frac{1}{4}}(x(0), y(0)) \tag{11}$$

and $\kappa_\delta = \{\gamma_2(\alpha - \beta - M), \gamma_1\alpha(\beta - M), \frac{1}{2}\gamma_2, \frac{1}{2}\gamma_1(\beta - M)\}$.

In order to present the information in a better way, the proof of the previous theorems will be presented in the next section. Indeed, a more general case for the twisting algorithm will be considered: a linear compensator is added to the twisting algorithm so linear increasing perturbations can be compensated, and the performance of this algorithm is improved.

III. STABILITY ANALYSIS FOR THE UNPERTURBED DYNAMICS USING A LINEAR COMPENSATOR.

Consider the "twisting" algorithm, but this time a linear compensator will be used (see (Orlov, 2006)),

$$u = -\alpha \text{sgn}(x) - \beta \text{sgn}(y) - hx - py \tag{12}$$

where $\alpha, \beta, h, p > 0$ are control parameters and consider $\delta = 0$. The system (12) is globally uniformly finite time stable if the inequality

$$\alpha > \beta > 0 \tag{13}$$

holds (see (Orlov, 2005)). Consider that the right hand side of (4) is discontinuous, so the solutions of the system (4) are defined in Filippov sense. Let

$$W(x,y) = V(x,y)+\frac{1}{2}h\gamma_1 x^2y^2+\frac{1}{4}h^2\gamma_1 x^4+h\alpha\gamma_1|x|^3 + \frac{2}{5}p\gamma_2|x|^{\frac{5}{2}} \quad (14)$$

where $V(x,y)$ is described by (6), with $\gamma_i > 0$. This function is continuous everywhere but not differentiable for $x = 0$. It will be shown that (14) is a strict Lyapunov function for the unperturbed system (12), so the finite time stability analysis will be proved. Moreover, an upper bound for time convergence, of the trajectories of (12) to the equilibrium point $(x,y) = (0,0)$, can be calculated from this analysis.

Theorem 3: Let (14) a strict Lyapunov function for the unperturbed system (12), where

$$\gamma_2 < \frac{4\sqrt{2}}{3}\gamma_1\beta\sqrt{\alpha} \quad (15)$$

holds for all $\gamma_i > 0$, $i = 1,2$, and let the parameters α and β from the system (12) be such that the condition (13) is satisfied. Then the system (12) is global finite time stable around the origin with

$$t_{reach} \leq \frac{2}{aW(x_0,y_0)^{\frac{1}{8}}}+\frac{2\sqrt{b}}{a^{\frac{3}{2}}}\tan^{-1}\left(\frac{\sqrt{b}W(x_0,y_0)^{\frac{1}{8}}}{\sqrt{a}}\right) \quad (16)$$

and where $a = \frac{min(\kappa)}{3\lambda_{max}^{\frac{3}{4}}(\Theta)}$, $b = \frac{1}{2}\frac{min(h,p)}{\gamma_1}$ and $\kappa = \{\gamma_2(\alpha-\beta), \gamma_1\alpha\beta, \frac{1}{2}\gamma_2, \frac{1}{2}\gamma_1\beta\}$. as an upper bound for time convergence for the trajectories to the equilibrium point.
proof: In order to show that $W(x,y)$ is a positive definite Lyapunov function, it can be described as follows

$$W(x,y) = \gamma_1 V_e^2(x,y)+\gamma_2|x|^{\frac{3}{2}}\text{sgn}(x)y+\frac{2}{5}p\gamma_2|x|^{\frac{5}{2}} \quad (17)$$

where the energy of the system (4) is described by $V_e(x,y) = (\alpha+\frac{1}{2}h|x|)|x|+\frac{1}{2}y^2$. Now consider $\tilde{\alpha} = \alpha+\frac{1}{2}h|x|$ and $\gamma_2 = \gamma_1\gamma_b$, using quadratic forms, the function (14) can be written as follows

$$W(x,y) = \frac{1}{2}\gamma_1|x|\rho^T P\rho+\frac{1}{4}\gamma_1 y^4+\frac{2}{5}p\gamma_2|x|^{\frac{5}{2}} \quad (18)$$

where $\rho^T = \left[|x|^{\frac{1}{2}}\text{sgn}(x) \;\; y\right]$, and

$$P = \begin{pmatrix} 2\tilde{\alpha}^2 & \gamma_b \\ \gamma_b & 2\tilde{\alpha}\end{pmatrix} \geq \begin{pmatrix} 2\alpha^2 & \gamma_b \\ \gamma_b & 2\alpha\end{pmatrix}. \quad (19)$$

So the function (6) is positive definite if $det(P) \geq 4\alpha^3 - \gamma_2^2 > 0$ holds. Now, consider an upper bound for the Lyapunov function (18) governed by

$$W(x,y) \leq \lambda_{max}(P)\|\rho\|+\frac{1}{4}\gamma_4 y^4 \quad (20)$$

where $\rho^T = \left[|x|^{\frac{1}{2}}\text{sgn}(x) \;\; |y|\right]$. Then, equation (20) can be written as follows

$$W(x,y) \leq \rho_1^T\Theta\rho_1 \leq \lambda_{max}(\Theta)\left(|x|^{\frac{1}{2}}+|y|\right)^4 \quad (21)$$

where $\rho_1^T = \left[|x| \;\; |y|^2\right]$, and

$$\Theta = \begin{pmatrix} \lambda_{max}(P) & \frac{1}{2}\lambda_{max}(P) \\ \frac{1}{2}\lambda_{max}(P) & \frac{1}{4}\gamma_4\end{pmatrix} \quad (22)$$

as a positive definite matrix. The time derivative of (14) along the trajectories of the system (12) is given by

$$\begin{aligned}\dot{W}(x,y) =\; & 2\alpha^2\gamma_1 xy+\frac{3}{2}\gamma_2 y^2|x|^{\frac{1}{2}} \\ +\; & \gamma_2|x|^{\frac{3}{2}}\left(-\alpha-\beta\text{sgn}(xy)-h|x|-p\text{sgn}(x)y\right) \\ +\; & 2\alpha\gamma_1|x|y\left(-\alpha\text{sgn}(x)-\beta\text{sgn}(y)-hx-py\right) \\ +\; & \alpha\gamma_1 y^2\text{sgn}(x)y+h\gamma_1 y^2 xy \\ +\; & \gamma_1 y^3\left(-\alpha\text{sgn}(x)-\beta\text{sgn}(y)-hx-py\right) \\ +\; & h\gamma_1 x^2 y\left(-\alpha\text{sgn}(x)-\beta\text{sgn}(y)-hx-py\right) \\ +\; & h^2\gamma_1 x^3 y+3h\alpha\gamma_1 x^2\text{sgn}(x)y+p\gamma_2|x|^{\frac{3}{2}}\text{sgn}(x)y\end{aligned}$$

after some algebraic simplifications,

$$\begin{aligned}\dot{W}(x,y) =\; & -\gamma_2|x|^{\frac{3}{2}}(\alpha-\beta)-\gamma_2 h|x|^{\frac{5}{2}}+\frac{3}{2}\gamma_2 y^2|x|^{\frac{1}{2}} \\ -\; & 2\alpha\beta\gamma_1|x||y|-2p\alpha\gamma_1|x|y^2-\beta\gamma_1|y|^3 \\ -\; & h\beta\gamma_1 x^2|y|-hp\gamma_1 x^2y^2-p\gamma_1 y^4\end{aligned} \quad (23)$$

In order to show that $\dot{W}(x,y) \leq 0$, using $\tilde{\beta} = (\beta+p|y|)$ this function will be written as follows

$$W(x,y) = -|y|\zeta^T Q\zeta-\gamma_2(\alpha+h|x|+\beta\text{sgn}(xy))|x|^{\frac{3}{2}} \quad (24)$$

where $\zeta^T = [|x|^{\frac{1}{2}} \;\; y]$

$$Q = \begin{pmatrix} \gamma_1(2\alpha+h|x|)\tilde{\beta} & -\frac{3}{2}\gamma_2 \\ -\frac{3}{2}\gamma_2 & \gamma_1\tilde{\beta}\end{pmatrix} \geq \begin{pmatrix} 2\gamma_1\alpha\beta & -\frac{3}{2}\gamma_2 \\ -\frac{3}{2}\gamma_2 & \gamma_1\beta\end{pmatrix} \quad (25)$$

where $det(Q) \geq 8\gamma_1^2\alpha\beta^2-\frac{9}{4}\gamma_2^2 > 0$ holds. In order to show the stability of the system (31), consider the following inequalities

$$W(x,y) \leq \lambda_{max}(\Theta)\left(|x|^{\frac{1}{2}}+|y|\right)^4+\gamma_1\left(x^2+y^2\right)^2 \quad (26)$$

$$\dot{W}(x,y) \leq -\frac{1}{3}min(\kappa)\left(|x|^{\frac{1}{2}}+|y|\right)^3-\frac{1}{2}\phi\left(x^2+y^2\right)^2 \quad (27)$$

where $\phi = min(h,p)$. Then, equation (26) and (27) can be written as follows

$$\dot{W}(x,y) \leq -\frac{min(\kappa)}{3\lambda_{max}^{\frac{3}{4}}(\Theta)}W(x,y)^{\frac{3}{4}}-\frac{1}{2}\frac{\phi}{\gamma_1}W(x,y) \quad (28)$$

where $\kappa = \{\gamma_2(\alpha-\beta), \gamma_1\alpha\beta, \frac{1}{2}\gamma_2, \frac{1}{2}\gamma_1\beta\}$. Consider the following comparison system

$$\dot{\omega} = -a\omega^{\frac{3}{4}}-b\omega \quad (29)$$

where $a = \frac{min(\kappa)}{3\lambda_{max}^{\frac{3}{4}}(\Theta)}$ and $b = \frac{1}{2}\frac{min(h,p)}{\gamma_1}$ and $\kappa_\delta = \{\gamma_2(\alpha-\beta), \gamma_1\alpha\beta, \frac{1}{2}\gamma_2, \frac{1}{2}\gamma_1\beta\}$. Using integral formulas an upper bound for reaching time can be obtained

$$t_{reach} \leq \frac{2}{a\omega^{\frac{1}{8}}} + \frac{2\sqrt{b}}{a^{\frac{3}{2}}} \tan^{-1}\left(\frac{\omega^{\frac{1}{8}}}{\sqrt{a1}}\right). \quad (30)$$

□

A. The perturbed dynamics using a linear compensator.

Consider the "twisting" algorithm, but this time a linear compensator will be used (see (Orlov, 2006)), so external linear increasing perturbations can be compensated

$$u = -\alpha \text{sgn}(x) - \beta \text{sgn}(y) - hx - py \quad (31)$$

where $\alpha, \beta, h, p > 0$ are control parameters and $|\delta| \leq M$ is an external bounded perturbation. It is well known that if If

$$\alpha - M > \beta > M \quad (32)$$

then the system (31) is globally uniformly finite time stable (see (Orlov, 2005)). Let

$$\begin{aligned} W(x,y) &= V(x,y) + \frac{1}{2}h\gamma_1 x^2 y^2 + \frac{1}{4}h^2\gamma_1 x^4 + h\alpha\gamma_1|x|^3 \\ &+ \frac{2}{5}p\gamma_2|x|^{\frac{5}{2}} \end{aligned} \quad (33)$$

where $V(x,y)$ is described by (6), with $\gamma_i > 0$. This function is continuous everywhere but not differentiable for $x = 0$. It will be shown that (33) is a strict Lyapunov function (see (Bacciotti, A., and Rosier, 2001)) for the perturbed system (31), so the finite time stability analysis for twisting algorithm will be proved. More over, an upper local bound for time convergence, of the trajectories of (31) to the equilibrium point $(x, y) = (0, 0)$, can be calculated from this analysis.

Theorem 4: Let (33) a strict Lyapunov function for the perturbed system (31), where

$$\gamma_2 < \frac{4\sqrt{2}}{3}\gamma_1(\beta - M)\sqrt{\alpha} \quad (34)$$

holds for all $\gamma_i > 0$, $i = 1, 2$, and let the parameters α and β be such that the condition (32) is satisfied. Then the system (31) is global finite time stable around the origin with

$$t_{reach} \leq \frac{2}{aW(x_0,y_0)^{\frac{1}{8}}} + \frac{2\sqrt{b}}{a^{\frac{3}{2}}} \tan^{-1}\left(\frac{\sqrt{b}W(x_0,y_0)^{\frac{1}{8}}}{\sqrt{a}}\right) \quad (35)$$

where $a = \frac{min(\kappa)}{3\lambda_{max}^{\frac{3}{4}}(\Theta)}$, $b = \frac{1}{2}\frac{min(h,p)}{\gamma_1}$ and $\kappa_\delta = \{\gamma_2(\alpha - \beta - M), \gamma_1\alpha(\beta - M), \frac{1}{2}\gamma_2, \frac{1}{2}\gamma_1(\beta - M)\}$ as an upper bound for time convergence for the trajectories to the equilibrium point.

proof: The time derivative of (33) along the trajectories of the system (31) is given by

$$\begin{aligned} \dot{W}(x,y) &= 2\gamma_1 xy + \frac{3}{2}\gamma_2 y^2|x|^{\frac{1}{2}} + \alpha\gamma_1 y^2 \text{sgn}(x)y + h\gamma_1 y^2 xy \\ &+ \gamma_2|x|^{\frac{3}{2}}\left(-\alpha - \beta\text{sgn}(xy) - h|x| - p\text{sgn}(x)y + M\right) \\ &+ 2\alpha\gamma_1|x|y\left(-\alpha\text{sgn}(x) - \beta\text{sgn}(y) - hx - py + M\right) \\ &+ \gamma_1 y^3\left(-\alpha\text{sgn}(x) - \beta\text{sgn}(y) - hx - py + M\right) \\ &+ h\gamma_1 x^2 y\left(-\alpha\text{sgn}(x) - \beta\text{sgn}(y) - hx - py + M\right) \\ &+ h^2\gamma_1 x^3 y + 3h\alpha\gamma_1 x^2 \text{sgn}(x)y + p\gamma_2|x|^{\frac{3}{2}}\text{sgn}(x)y \end{aligned}$$

after some algebraic simplifications,

$$\begin{aligned} \dot{W}(x,y) &\leq -\gamma_2|x|^{\frac{3}{2}}(\alpha + h|x| - \beta - M) \\ &+ \frac{3}{2}\gamma_2 y^2|x|^{\frac{1}{2}} - 2\alpha\gamma_1|x||y|(\beta + p|y| - M) \\ &- \gamma_1|y|^3(\beta + p|y| - M) - h\gamma_1 x^2|y|(\beta + p|y| - M) \end{aligned} \quad (36)$$

If $\alpha + h|x| > \beta + M$ and $\beta + p|y| > M$, it is easy to see that $\dot{W}(x,y) \leq 0$. Notice that the linear compensator is very useful when linear increasing perturbations are present, but near the origin, this linear part has almost no effect. Then, equation (36) can be written as follows

$$\dot{W}(x,y) \leq -\frac{min(\kappa)}{3\lambda_{max}^{\frac{3}{4}}(\Theta)}W(x,y)^{\frac{3}{4}} - \frac{min(h,p)}{\gamma_1}W(x,y) \quad (37)$$

Consider the following comparison system

$$\dot{\omega} = -a\omega^{\frac{3}{4}} - b\omega \quad (38)$$

where $a = \frac{min(\kappa)}{3\lambda_{max}^{\frac{3}{4}}(\Theta)}$ and $b = \frac{1}{2}\frac{min(h,p)}{\gamma_1}$. Using integral formulas an upper bound for reaching time can be obtained

$$t_{reach} \leq \frac{2}{a\omega(0)^{\frac{1}{8}}} + \frac{2\sqrt{b}}{a^{\frac{3}{2}}} \tan^{-1}\left(\frac{\omega(0)^{\frac{1}{8}}}{\sqrt{a1}}\right) \quad (39)$$

with $\kappa_\delta = \{\gamma_2(\alpha - \beta - M), \gamma_1\alpha(\beta - M), \frac{1}{2}\gamma_2, \frac{1}{2}\gamma_1(\beta - M)\}$.
□

IV. Numerical Experiments.

A. Problem Statement.

For this section, to stabilize the cart-pendulum system around its unstable equilibrium point, is considered. Some advantages of this work with respect (Riachy, Orlov, Floquet, Santiesteban and Jean-Pierre, 2008) can be shown, for example: an optimization can be used for the controller gains in order to improve the performance of the controller. The cart-pendulum equations are governed by

$$\begin{aligned} (M+m)\ddot{x} &+ ml\sin\theta\,\dot{\theta}^2 - ml\cos\theta\,\ddot{\theta} = \tau + w_1(t) - \psi(\dot{x}) \\ \frac{4}{3}ml^2\ddot{\theta} &- ml\cos\theta\,\ddot{x} - mgl\sin\theta = w_2(t) - \varphi(\dot{\theta}) \end{aligned}$$

where x is the cart position, θ is the angular deviation of the pendulum from the vertical, M is the cart mass, m is the rod mass, l is the distance to the center of mass of the pendulum, g is the gravitational acceleration, τ is the controlled input, $w_1(t), w_2(t)$ are external disturbances, $\psi(\dot{x})$ and $\varphi(\dot{\theta})$ are friction forces, affecting the cart and the pendulum, respectively. In order to describe the friction forces the classical model is used

$$\psi(\dot{x}) = \psi_v\dot{x} + \psi_c sign(\dot{x}), \;\; \varphi(\dot{\theta}) = \varphi_v\dot{\theta} + \varphi_c sign(\dot{\theta}). \quad (40)$$

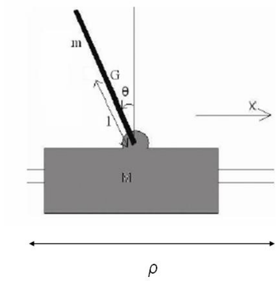

Figura 1. System cart-pendulum.

The above model comes with the viscous friction coefficients $\psi_v, \varphi_v > 0$, the Coulomb friction level $\psi_c, \varphi_c > 0$, and the standard notation $sign(\cdot)$ for the signum function. Upper bounds $N_i > 0$, $i = 1, 2$ for the magnitudes of uncertainty terms $w_1(t), w_2(t)$ can normally be estimated a priori:

$$|w_i(t)| \leq N_i \tag{41}$$

for all t. The *control objective* is as follows: a sliding mode controller is design to locally stabilize the cart-pendulum system around its unstable equilibrium point. The key idea is to find a diffeomorphic state space transformation to bring the cart-pendulum system into its regular form (see (Riachy, Orlov, Floquet, Santiesteban and Jean-Pierre, 2008) for details). For this purpose, let us first define the following local change of variable:

$$\eta = x - \frac{4}{3} l \varrho(\theta) \tag{42}$$

with

$$\varrho(\theta) = \ln\left(\frac{1+\sin\theta}{\cos\theta}\right), \quad |\theta| < \frac{\pi}{2}. \tag{43}$$

Now a fictitious output ξ is chosen to ensure that the $(\eta, \dot{\eta})$ subsystem with $\varphi = \omega_2 = 0$ is minimum phase with respect to this output. The required output ξ can be locally chosen as:

$$\xi = \tan\theta - \lambda_1 \eta - \lambda_2 \dot{\eta}, \tag{44}$$

with λ_1 and $\lambda_2 > 0$. Let us set the control law u as follows:

$$u = -\mu\left(\theta, \dot{\theta}\right) - \alpha_1 sign(\xi) - \beta_1 sign(\dot{\xi}),$$

which gives:

$$\tau = \frac{mlD\cos\theta}{\left[3ml + 8ml^2\lambda_2\dot{\theta}\sin\theta - 3\lambda_2\cos\theta\varphi_v\right]} \left(-\mu\left(\theta,\dot{\theta}\right) - \alpha_1(\xi,\dot{\xi})sign(\xi) - \beta_1(\xi,\dot{\xi})sign(\dot{\xi})\right) \tag{45}$$

with

$$\begin{aligned} \mu\left(\theta,\dot{\theta}\right) = {} & 2\frac{\tan\theta}{\cos^2\theta}\dot{\theta}^2 + \left[\frac{1}{\cos^2\theta} + \frac{8l\lambda_2\dot{\theta}\tan\theta}{3\cos\theta} - \frac{\lambda_2\varphi_v}{ml\cos\theta}\right] \\ & \left[\frac{3[(M_c+m)g - ml\cos\theta\,\dot{\theta}^2]\sin\theta - 3\cos\theta\psi(\dot{x}) - 3\frac{M_c+m}{ml}\varphi(\dot{\theta})}{D}\right] \\ & + \left[g + \frac{4}{3}\frac{l\dot{\theta}^2}{\cos\theta}\right]\lambda_1\tan\theta + \left[g + \frac{4}{3}\frac{l\dot{\theta}^2(1+\sin^2\theta)}{\cos\theta}\right]\lambda_2\frac{\dot{\theta}}{\cos^2\theta} \\ & - \left[\frac{\lambda_1 + \lambda_2\dot{\theta}\tan\theta}{ml\cos\theta}\right]\varphi(\dot{\theta}). \end{aligned}$$

We considered the real parameters of the laboratory cart-pendulum system from (Riachy, Orlov, Floquet, Santiesteban and Jean-Pierre, 2008). These parameters are listed in Table 1. The initial conditions of the position of the cart-pendulum system and that of the modified Van der Pol oscillator, selected for all experiments, were $x(0) = -0.07$, $\theta(0) = 0.01\ rad$, whereas all the velocity initial conditions were set to zero. The gains are fixed as $\alpha = 9$, $\beta = 5$, $h = 2$, $p = 2$. The external disturbances $w_1(t) = 0.5 + 0.5sin(t)\ N$, $w_2(t) \equiv 0.0001 + 0.0001sin(t)\ N \cdot m$, **Notice** that the gains are modified so the conditions for finite time stability (32) are satisfied. The first graph in figure 3 shows the dynamics of cart-pendulum system in closed loop. The third graph in figure 3 shows the applied torque.

TABLA I

PARAMETERS OF THE CART-PENDULUM.

Notation	Value	Units
M	3.4	kg
m	0.147	kg
l	0.175	m
ψ_v	8.5	$N \cdot s/m$
φ_v	0.0015	$N \cdot m \cdot s/rad$
ψ_c	6.5	N
φ_c	0.00115	$N \cdot m$

V. CONCLUSIONS

In this paper, a strict Lyapunov function to prove global finite-time stability of the twisting algorithm is proposed. Moreover, an upper bound of time convergence for non perturbed and perturbed twisting algorithm has been estimated. The stability proof for twisting algorithm with a linear compensator has been shown, so linear increasing perturbations can be compensated and the controller gains can be optimized using this kind of algorithm. Based on the proposed strict Lyapunov function, a local upper bound for time convergence, of the trajectories of the system (12) to its equilibrium point, has been calculated. The stability analysis is reinforced with numerical experiments showing the results of this work. More over, an optimization of the controller gain is used so the performance of the quasi-homogeneous synthesis is improved.

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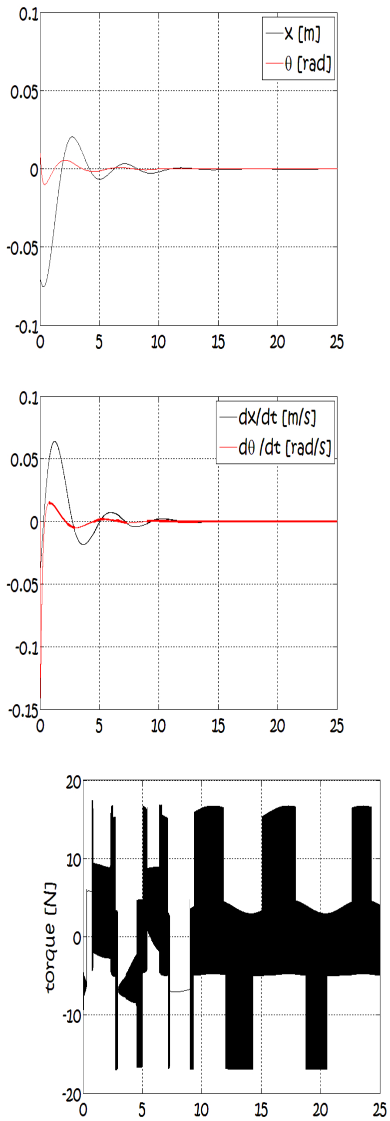

Figura 2. Orbital stabilization of the cart-pendulum system

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